2.4 Multipole expansion

In the last section we encountered a number of axisymmetric density distributions that give rise to potentials of known form. By adding a few of these distributions together one can obtain quite a wide range of model galaxies that have readily available potentials. However, for many purposes one requires a systematic procedure for calculating the potential of an arbitrary density distribution to whatever accuracy one pleases. The next few sections are devoted to this task.

The first such technique, based on spherical harmonics, works best for systems that are neither very flattened nor very elongated. Hence it is a good method for calculating the potentials of bulges and dark-matter halos (§§1.1.2 and 1.1.3).

Our first step is to obtain the potential of a thin spherical shell of variable surface density. Since the shell has negligible thickness, the task of solving Poisson's equation $\nabla^2\Phi = 4\pi G\rho$ reduces to that of solving Laplace's equation $\nabla^2\Phi = 0$ inside and outside the shell, subject to suitable boundary conditions at infinity, at the origin, and on the shell. Now in spherical coordinates Laplace's equation is (eq. B.53)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0. \tag{2.76}$$

This may be solved by the method of **separation of variables**. We seek special solutions that are the product of functions of one variable only:

$$\Phi(r,\theta,\phi)=R(r)P(\theta)Q(\phi). \tag{2.77a}$$

Substituting equation (2.77a) into (2.76) and rearranging, we obtain

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)=-\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}. \tag{2.77b}$$

The left side of this equation does not depend on ϕ, and the right side does not depend on r or θ. It follows that both sides are equal to some constant, say m^2. Hence

$$-\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}=m^2, \tag{2.78a}$$

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)=m^2. \tag{2.78b}$$

Equation (2.78a) may be immediately integrated to

$$Q(\phi)=Q_m^+\mathrm{e}^{im\phi}+Q_m^-\mathrm{e}^{-im\phi}. \tag{2.79a}$$

We require Φ to be a periodic function of ϕ with period 2π, so m can take only integer values. Since equations (2.78) depend only on m^2, we could restrict our attention to non-negative values of m without loss of generality. However, a simpler convention is to allow m to take both positive and negative values, so the second exponential in equation (2.79a) becomes redundant, and we may write simply

$$Q = Q_m \mathrm{e}^{\mathrm{i}m\phi} \qquad (m = \ldots, -1, 0, 1, \ldots). \tag{2.79b}$$

Equation (2.78b) can be written

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \frac{m^2}{\sin^2\theta} - \frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right). \tag{2.80}$$

Since the left side of this equation does not depend on θ and the right side does not depend on r, both sides must equal some constant, which we write as $l(l+1)$. Thus equation (2.80) implies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0, \tag{2.81a}$$

and in terms of $x \equiv \cos\theta$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right] - \frac{m^2}{1-x^2}P + l(l+1)P = 0. \tag{2.81b}$$

Two linearly independent solutions of equation (2.81a) are

$$R(r) = Ar^l \quad \text{and} \quad R(r) = Br^{-(l+1)}. \tag{2.82}$$

The solutions of equation (2.81b) are associated Legendre functions $P_l^m(x)$ (see Appendix C.5). Physically acceptable solutions exist only when l is an integer. Without loss of generality we can take l to be non-negative, and then physically acceptable solutions exist only for $|m| \le l$. When $m = 0$ the solutions are simply polynomials in x, called Legendre polynomials $P_l(x)$.

Rather than write out the product $P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ again and again, it is helpful to define the spherical harmonic $\mathrm{Y}_l^m(\theta,\phi)$, which is equal to $P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ times a constant chosen so the Y_l^m satisfy the orthogonality relation (see eq. C.44)

$$\begin{aligned}\int \mathrm{d}^2\Omega\, \mathrm{Y}_l^{m*}(\mathbf{\Omega})\mathrm{Y}_{l'}^{m'}(\mathbf{\Omega}) &\equiv \int_0^\pi \mathrm{d}\theta\,\sin\theta \int_0^{2\pi}\mathrm{d}\phi\, \mathrm{Y}_l^{m*}(\theta,\phi)\mathrm{Y}_{l'}^{m'}(\theta,\phi) \\ &= \delta_{ll'}\delta_{mm'},\end{aligned} \tag{2.83}$$

where we have used $\mathbf{\Omega}$ as a shorthand for (θ,ϕ) and $\mathrm{d}^2\Omega$ for $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. The spherical harmonics with $l \le 2$ are listed in equation (C.50).

Putting all these results together, we have from equations (2.77a), (2.79b), and (2.82) that

$$\Phi_{lm}(r, \mathbf{\Omega}) = \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) \mathrm{Y}_l^m(\mathbf{\Omega}) \tag{2.84}$$

is a solution of $\nabla^2\Phi = 0$ for all non-negative integers l and integer m in the range $-l \le m \le l$.

Now let us apply these results to the problem of determining the potential of a thin shell of radius a and surface density $\sigma(\mathbf{\Omega})$. We write the potential internal and external to the shell as

$$\Phi_{\rm int}(r, \mathbf{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) \mathrm{Y}_l^m(\mathbf{\Omega}) \quad (r \le a), \tag{2.85a}$$

and

$$\Phi_{\rm ext}(r, \mathbf{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(C_{lm} r^l + D_{lm} r^{-(l+1)}\right) \mathrm{Y}_l^m(\mathbf{\Omega}) \quad (r \ge a). \tag{2.85b}$$

The potential at the center must be non-singular, so $B_{lm} = 0$ for all l, m. Similarly, the potential at infinity must be zero, so $C_{lm} = 0$ for all l, m. Furthermore, $\Phi_{\rm ext}(a, \mathbf{\Omega})$ must equal $\Phi_{\rm int}(a, \mathbf{\Omega})$ because no work can be done in passing through an infinitesimally thin shell. Hence from equations (2.85) we have

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l \mathrm{Y}_l^m(\mathbf{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} D_{lm} a^{-(l+1)} \mathrm{Y}_l^m(\mathbf{\Omega}). \tag{2.86}$$

The coefficients $A_{lm} a^l$ etc. of each spherical harmonic Y_l^m on the two sides of equation (2.86) must be equal, as can be shown by multiplying both sides of the equation by $\mathrm{Y}_{l'}^{m'*}(\mathbf{\Omega})$, integrating over $\mathbf{\Omega}$, and using the orthogonality relation (2.83). Therefore, from equation (2.86) we have

$$D_{lm} = A_{lm} a^{2l+1}. \tag{2.87}$$

Now let us expand the surface density of the thin shell as

$$\sigma(\mathbf{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} \mathrm{Y}_l^m(\mathbf{\Omega}), \tag{2.88}$$

where the σ_{lm} are numbers yet to be determined. To obtain the coefficient $\sigma_{l'm'}$, we multiply both sides of equation (2.88) by $\mathrm{Y}_{l'}^{m'*}(\mathbf{\Omega})$ and integrate over $\mathbf{\Omega}$. With equation (2.83) we find

$$\int \mathrm{d}^2\Omega \, \mathrm{Y}_{l'}^{m'*}(\mathbf{\Omega}) \, \sigma(\mathbf{\Omega}) = \sigma_{l'm'}. \tag{2.89}$$

Since $\mathrm{Y}_0^0 = 1/\sqrt{4\pi}$, $\sigma_{00} = M/(2a^2\sqrt{\pi})$, where M is the mass of the shell.

Gauss's theorem (2.12) applied to a small piece of the shell tells us that

$$\left(\frac{\partial\Phi_{\rm ext}}{\partial r}\right)_{r=a} - \left(\frac{\partial\Phi_{\rm int}}{\partial r}\right)_{r=a} = 4\pi G\sigma(\mathbf{\Omega}), \tag{2.90}$$

so inserting equations (2.85) and (2.88) into equation (2.90), we obtain

$$\begin{aligned} -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big((l+1)D_{lm}a^{-(l+2)}+lA_{lm}a^{l-1}\Big)\mathrm{Y}_l^m(\mathbf{\Omega}) = \\ 4\pi G\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sigma_{lm}\mathrm{Y}_l^m(\mathbf{\Omega}). \end{aligned} \tag{2.91}$$

Once again the coefficients of Y_l^m on each side of the equation must be identical, so with (2.87) we have

$$A_{lm} = -4\pi Ga^{-(l-1)}\frac{\sigma_{lm}}{2l+1} \quad ; \quad D_{lm} = -4\pi Ga^{l+2}\frac{\sigma_{lm}}{2l+1}. \tag{2.92}$$

Collecting these results together, we have from equations (2.85) that

$$\begin{aligned} \Phi_{\rm int}(r,\mathbf{\Omega}) &= -4\pi Ga\sum_{l=0}^{\infty}\left(\frac{r}{a}\right)^l\sum_{m=-l}^{l}\frac{\sigma_{lm}}{2l+1}\mathrm{Y}_l^m(\mathbf{\Omega}), \\ \Phi_{\rm ext}(r,\mathbf{\Omega}) &= -4\pi Ga\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^{l+1}\sum_{m=-l}^{l}\frac{\sigma_{lm}}{2l+1}\mathrm{Y}_l^m(\mathbf{\Omega}), \end{aligned} \tag{2.93}$$

where the σ_{lm} are given by equation (2.89).

Finally we evaluate the potential of a solid body by breaking it down into a series of spherical shells. We let $\delta\sigma_{lm}(a)$ be the σ-coefficient of the shell lying between a and $a+\delta a$, and $\delta\Phi(r,\mathbf{\Omega};a)$ be the corresponding potential at r. Then we have by equation (2.89)

$$\delta\sigma_{lm}(a) = \int_0^{\pi}\mathrm{d}\theta\,\sin\theta\int_0^{2\pi}\mathrm{d}\phi\,\mathrm{Y}_l^{m*}(\mathbf{\Omega})\rho(a,\mathbf{\Omega})\delta a \equiv \rho_{lm}(a)\delta a. \tag{2.94}$$

Substituting these values of σ_{lm} into equations (2.93) and integrating over all a, we obtain the potential at r generated by the entire collection of shells:

$$\begin{aligned} \Phi(r,\mathbf{\Omega}) &= \sum_{a=0}^{r}\delta\Phi_{\rm ext} + \sum_{a=r}^{\infty}\delta\Phi_{\rm int} \\ &= -4\pi G\sum_{l,m}\frac{\mathrm{Y}_l^m(\mathbf{\Omega})}{2l+1}\left(\frac{1}{r^{l+1}}\int_0^r\mathrm{d}a\,a^{l+2}\rho_{lm}(a) + r^l\int_r^{\infty}\frac{\mathrm{d}a}{a^{l-1}}\rho_{lm}(a)\right). \end{aligned} \tag{2.95}$$

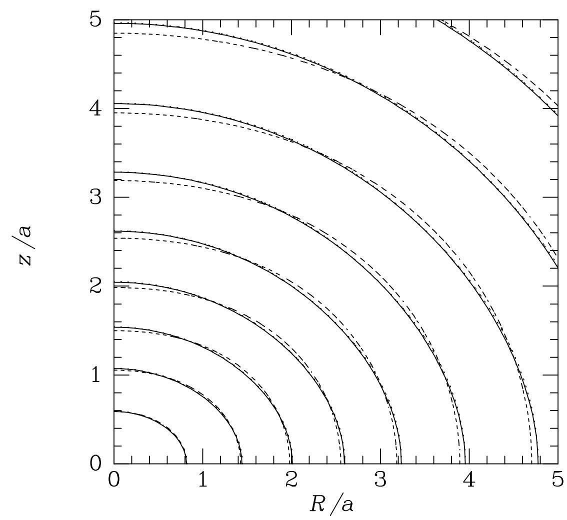

Figure 2.10 Equipotentials of Satoh's density distribution (2.70c) with $b/a = 1$. Full curves show the exact equipotentials computed from equation (2.70a), and dashed curves show the estimate provided by equation (2.95) with the sum over l extending to $l = 2$. Contours based on the sum to $l = 8$ are also plotted (dotted contours) but almost overlie the full curves.

This equation gives the potential generated by the body as an expansion in **multipoles**: the terms associated with $l = m = 0$ are the **monopole** terms, those associated with $l = 1$ are **dipole** terms, those with $l = 2$ are **quadrupole** terms, and those with larger l are 2^l-poles. Similar expansions occur in electrostatics (e.g., Jackson 1999). The monopole terms are the same as in equation (2.28) for the potential of a spherical system.[6] Since there is no gravitational analog of negative charge, pure dipole or quadrupole gravitational potentials cannot arise, in contrast to the electrostatic case. In fact, if one places the origin of coordinates at the center of mass of the system, the dipole term vanishes identically outside any matter distribution. While the monopole terms generate a circular-speed curve $v_c(r) = \sqrt{GM(r)/r}$ that never declines with increasing r more steeply than in the Keplerian case ($v_c \propto r^{-1/2}$), over a limited range in r the higher-order multipoles may cause the circular speed to fall more steeply with increasing radius.

As an illustration of the effectiveness of the multipole expansion, we show in Figure 2.10 the contours of Satoh's potential $\Phi_S(R, z)$ (eq. 2.70a), together with the approximations to this potential that one obtains from equation (2.95) if one includes only terms with $l \leq 2$ or 8. The flexibility

[6] Thus the spherical-harmonic expansion provides an alternative proof of Newton's first and second theorems.

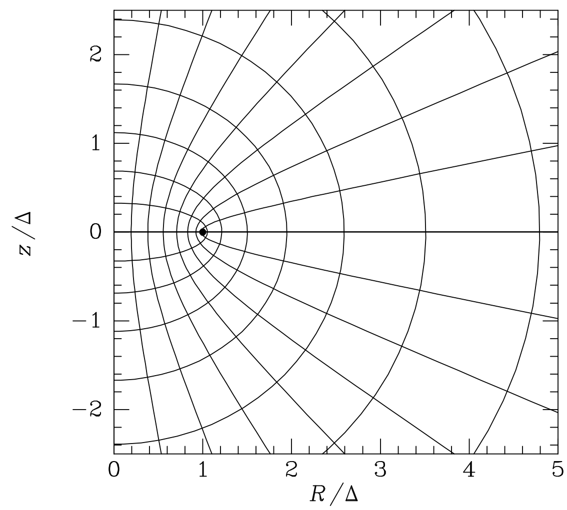

Figure 2.11 Curves of constant u and v in the (R, z) plane. Semi-ellipses are curves of constant u, and hyperbolae are curves of constant v. The common focus of all curves is marked by a dot. In order to ensure that each point has a unique v-coordinate, we exclude the disk $(z = 0,\ R \le \Delta)$ from the space to be considered.

of the multipole expansion makes it a powerful tool for numerical work, and it plays a central role in some of the Poisson solvers for N-body codes that will be described in §2.9. However, multipole expansions are poorly suited for modeling the potentials of disks.

2.5 The potentials of spheroidal and ellipsoidal systems

Many galaxies have nearly spheroidal or ellipsoidal equidensity surfaces (BM §4.3.3). Moreover, Newton's theorems for spherical bodies can be generalized to include spheroidal and ellipsoidal bodies, so models with isodensity surfaces of this shape are relatively easy to construct. Finally, as the axis ratio shrinks to zero a spheroid becomes a disk, and thus we can obtain the potentials of razor-thin disks as a limiting case of spheroids.

In this section we develop efficient techniques for calculating the potentials of such objects. In §§2.5.1 and 2.5.2 we derive formulae for oblate (i.e., flattened) spheroidal systems, and in §2.5.3 we briefly discuss ellipsoidal systems. Results for prolate spheroidal systems can be obtained either by adapting our derivations for oblate systems, or by specializing the results for ellipsoidal systems. The principal formulae for all three geometries are given in Tables 2.1 and 2.2.