C.5 Legendre functions

The Legendre functions of the first and second kinds, $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$, are linearly independent solutions of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}w}{\mathrm{d}z}\right] - \frac{\mu^2}{1-z^2}w + \lambda(\lambda+1)w = 0. \tag{C.30}$$

For $\mathrm{Re}(\lambda) > 0$, the Legendre functions of the first kind diverge ($\propto z^\lambda$) as $|z| \to \infty$, while the functions of the second kind vanish [$\propto z^{-(\lambda+1)}$]. As $z \to 0$

$$\begin{aligned}\left[\frac{d\ln P_\lambda^\mu(z)}{dz}\right]_{z=0} &= 2\tan[\tfrac{1}{2}\pi(\lambda+\mu)]\frac{[\frac{1}{2}(\lambda+\mu)]![\frac{1}{2}(\lambda-\mu)]!}{[\frac{1}{2}(\lambda+\mu-1)]![\frac{1}{2}(\lambda-\mu-1)]!},\\ \left[\frac{d\ln Q_\lambda^\mu(z)}{dz}\right]_{z=0} &= 2\exp\{\tfrac{1}{2}\pi\mathrm{i}\,\mathrm{sgn}[\mathrm{Im}(z)]\}\frac{[\frac{1}{2}(\lambda+\mu)]![\frac{1}{2}(\lambda-\mu)]!}{[\frac{1}{2}(\lambda+\mu-1)]![\frac{1}{2}(\lambda-\mu-1)]!}.\end{aligned} \tag{C.31}$$

Here $\mathrm{sgn}(x) = +1$ if $x > 0$ and -1 if $x < 0$.

For many applications we are interested in the Legendre functions with real arguments, $z = x$, in the interval $-1 \leq x \leq 1$. Unless μ is an even integer, $P_\lambda^\mu(x+\mathrm{i}\epsilon)$ and $P_\lambda^\mu(x-\mathrm{i}\epsilon)$ are different for real x and ϵ as $\epsilon \to 0$. Thus it is conventional to redefine the Legendre functions for $-1 \leq x \leq 1$ by

$$\begin{aligned}P_\lambda^\mu(x) &\equiv \tfrac{1}{2}\lim_{\epsilon\to 0}\left[\mathrm{e}^{\pi\mathrm{i}\mu/2}P_\lambda^\mu(x+\mathrm{i}|\epsilon|) + \mathrm{e}^{-\pi\mathrm{i}\mu/2}P_\lambda^\mu(x-\mathrm{i}|\epsilon|)\right]\\ Q_\lambda^\mu(x) &\equiv \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\mu}\lim_{\epsilon\to 0}\left[\mathrm{e}^{-\pi\mathrm{i}\mu/2}Q_\lambda^\mu(x+\mathrm{i}|\epsilon|) + \mathrm{e}^{\pi\mathrm{i}\mu/2}Q_\lambda^\mu(x-\mathrm{i}|\epsilon|)\right].\end{aligned} \tag{C.32}$$

For $\mu = 0$ and λ a non-negative integer, the Legendre functions are polynomials given by the formula

$$P_l(x) \equiv P_l^0(z) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l. \tag{C.33}$$

These **Legendre polynomials** are also generated by the relation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l \quad |t| < 1,\ |x| \leq 1, \tag{C.34}$$

which leads to an expression for the inverse distance between the points $\mathbf{x}$ and $\mathbf{x}'$,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma), \tag{C.35}$$

where $r_< = \min(|\mathbf{x}|, |\mathbf{x}'|)$, $r_> = \max(|\mathbf{x}|, |\mathbf{x}'|)$, and γ is the angle between the two vectors.

For integer $m > 0$ and integer $l \geq 0$ the Legendre functions are sometimes called **associated Legendre functions**, and are given by[1]

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{\mathrm{d}^m P_l^0(x)}{\mathrm{d}x^m} = (-1)^m\frac{(1-x^2)^{m/2}}{2^l l!}\frac{\mathrm{d}^{l+m}P_l^0(x)}{\mathrm{d}x^{l+m}}. \tag{C.36}$$

[1] Our convention for associated Legendre functions with real x between -1 and $+1$ follows Abramowitz & Stegun (1964), Press et al. (1986), Gradshteyn & Ryzhik (2000), and software such as IDL, Maple, and Mathematica, but differs from Morse & Feshbach (1953), Arfken & Weber (2005), and the first edition of this book by a factor $(-1)^m$.

Note that $P_l^m(x)$ vanishes for $m > l$, and that $P_l^m(x)$ is even in x if $l-m$ is even, and odd if $l-m$ is odd. We have

$$P_l^m(0) = (-1)^{(l+m)/2} \frac{(l+m)!}{2^l [\frac{1}{2}(l-m)]! \, [\frac{1}{2}(l+m)]} \quad (l-m \text{ even}), \tag{C.37}$$

and zero if $(l-m)$ is odd. For integer $m > 0$,

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \tag{C.38}$$

The associated Legendre functions are orthogonal in the sense that

$$\int_{-1}^{1} \mathrm{d}x \, P_l^m(x) P_n^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ln}. \tag{C.39}$$

$$\int_{-1}^{1} \frac{\mathrm{d}x}{1-x^2} \, P_l^m(x) P_l^k(x) = \frac{1}{m} \frac{(l+m)!}{(l-m)!} \delta_{mk}. \tag{C.40}$$

The associated Legendre functions can be written most compactly using the substitution $x = \cos\theta$; since $-1 \le x \le 1$ we take $0 \le \theta \le \pi$ and let $c = \cos\theta$, $s = \sin\theta$:

$$\begin{array}{llll} P_0(c) = 1 & & & \\ P_1(c) = c & P_1^1(c) = -s & & \\ P_2(c) = \frac{1}{2}(3c^2 - 1) & P_2^1(c) = -3cs & P_2^2(c) = 3s^2 & \\ P_3(c) = \frac{1}{2}(5c^3 - 3c) & P_3^1(c) = -\frac{3}{2}s(5c^2 - 1) & P_3^2(c) = 15cs^2 & P_3^3(c) = -15s^3. \end{array} \tag{C.41}$$

C.6 Spherical harmonics

A spherical harmonic is defined by the expression

$$\mathrm{Y}_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi} \quad (m \ge 0). \tag{C.42}$$

For most purposes, the indices of the spherical harmonics can be restricted to $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$. The variables lie in the range $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$ and usually represent the angular coordinates in a spherical coordinate system (see Figure B.1). Note that

$$\mathrm{Y}_l^{-m}(\theta, \phi) = (-1)^m \mathrm{Y}_l^{m*}(\theta, \phi), \tag{C.43}$$

where the asterisk denotes complex conjugation.

The most important feature of the spherical harmonics, which is easily proved using equation (C.39), is that they are orthonormal in the sense that

$$\oint \mathrm{d}^2\Omega \, \mathrm{Y}_k^{n*}(\mathbf{\Omega}) \mathrm{Y}_l^m(\mathbf{\Omega}) = \int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\phi \, \mathrm{Y}_k^{n*}(\theta, \phi) \mathrm{Y}_l^m(\theta, \phi) = \delta_{kl}\delta_{nm}. \tag{C.44}$$

An arbitrary function of position $f(\mathbf{r})$ can be written in spherical coordinates as a series of spherical harmonics,

$$f(\mathbf{r}) = f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r) \mathrm{Y}_l^m(\theta, \phi). \tag{C.45}$$

Multiplying by $\mathrm{Y}_n^{k\,*}(\theta,\phi)$, integrating over solid angle, and using equation (C.44), we find

$$f_{nk}(r) = \int \mathrm{d}^2\Omega\, \mathrm{Y}_k^{n\,*}(\theta,\phi) f(\mathbf{r}). \tag{C.46}$$

The addition theorem for spherical harmonics states that if the directions (θ,ϕ) and (θ',ϕ') are separated by an angle γ, then

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} \mathrm{Y}_l^{m\,*}(\theta',\phi')\mathrm{Y}_l^m(\theta,\phi). \tag{C.47}$$

Together with equation (C.35), this leads to an expression for the inverse distance between the points $\mathbf{x} = (r,\theta,\phi)$ and $\mathbf{x}' = (r',\theta',\phi')$:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\mathrm{Y}_l^{m\,*}(\theta',\phi')\mathrm{Y}_l^m(\theta,\phi), \tag{C.48}$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$.

Using equations (B.53) and (C.30) we can show that

$$\nabla^2[f(r)\mathrm{Y}_l^m(\theta,\phi)] = \left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2}f(r)\right]\mathrm{Y}_l^m(\theta,\phi). \tag{C.49}$$

The first few spherical harmonics are:

$$\begin{aligned}
&\mathrm{Y}_0^0(\theta,\phi) = \tfrac{1}{\sqrt{4\pi}} & & \\
&\mathrm{Y}_1^0(\theta,\phi) = \sqrt{\tfrac{3}{4\pi}}\cos\theta & &\mathrm{Y}_1^{\pm1}(\theta,\phi) = \mp\sqrt{\tfrac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} \\
&\mathrm{Y}_2^0(\theta,\phi) = \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1) & &\mathrm{Y}_2^{\pm1}(\theta,\phi) = \mp\sqrt{\tfrac{15}{8\pi}}\sin\theta\cos\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} \\
& & &\mathrm{Y}_2^{\pm2}(\theta,\phi) = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\,\mathrm{e}^{\pm2\mathrm{i}\phi}.
\end{aligned} \tag{C.50}$$

C.7 Bessel functions

The most complete reference is Watson (1995).

The Bessel functions of the first and second kind, $J_\nu(z)$ and $Y_\nu(z)$, are linearly independent solutions of the differential equation

$$\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}w}{\mathrm{d}z}\right) + \left(1 - \frac{\nu^2}{z^2}\right)w = 0. \tag{C.51}$$

In series form,

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(\nu+k)!}(\tfrac{1}{2}z)^{\nu+2k}, \tag{C.52}$$

and $Y_\nu(z)$ is defined by the relation

$$Y_\nu(z) = \frac{\cos\nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi}, \tag{C.53}$$

or by its limiting value if ν is an integer. The function $Y_\nu(z)$ diverges as $z^{-|\nu|}$ when $z \to 0$. As $x \to \infty$

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + \mathrm{O}(x^{-1}). \tag{C.54}$$

If $\nu \equiv n$ is an integer

$$J_n(-z) = (-1)^n J_n(z) \quad ; \quad J_{-n}(z) = (-1)^n J_n(z) \quad ; \quad Y_{-n}(z) = (-1)^n Y_n(z), \tag{C.55}$$

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \cos(z \sin\theta - n\theta). \tag{C.56}$$

If C_ν denotes either J_ν or Y_ν,

$$C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z) \quad ; \quad C_{\nu-1}(z) - C_{\nu+1}(z) = 2\frac{\mathrm{d}C_\nu(z)}{\mathrm{d}z}. \tag{C.57}$$

For $\nu = 0$ these relations imply

$$J_0'(z) = -J_1(z). \tag{C.58}$$

An important integral identity is

$$\int_0^\infty \mathrm{d}k\, k \int_0^\infty \mathrm{d}R\, R F(R) J_\nu(kR) J_\nu(kr) = F(r) \quad (\nu \geq -\tfrac{1}{2}), \tag{C.59}$$

where $F(R)$ is an arbitrary function. If

$$g(k) = \int_0^\infty \mathrm{d}r\, r f(r) J_\nu(kr) \tag{C.60a}$$

then g is called the **Hankel transform** of f, and equation (C.59) yields

$$f(r) = \int_0^\infty \mathrm{d}k\, k g(k) J_\nu(kr). \tag{C.60b}$$

The Hankel transform is defined for any integer ν and all real $\nu \geq -\frac{1}{2}$.

The **modified Bessel functions** are

$$I_\nu(z) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}z) \quad ; \quad K_\nu(z) = K_{-\nu}(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}; \tag{C.61}$$

the second equation is replaced by its limiting value if ν is an integer. As $z \to 0$,

$$\begin{aligned} I_\nu(z) &\to \frac{1}{\nu!}\left(\tfrac{1}{2}z\right)^\nu \quad (\nu \neq -1, -2, \ldots); \\ K_\nu(z) &\to \frac{(\nu-1)!}{2}\left(\tfrac{1}{2}z\right)^{-\nu} \quad (\nu > 0); \end{aligned} \tag{C.62}$$

At large x,

$$I_\nu(x) \to \frac{\mathrm{e}^x}{\sqrt{2\pi x}} \quad ; \quad K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}. \tag{C.63}$$

If $\nu \equiv n$ is an integer,

$$I_n(-z) = (-1)^n I_n(z) \quad ; \quad I_n(z) = I_{-n}(z) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \mathrm{e}^{z\cos\theta}\cos(n\theta). \tag{C.64}$$

If Z_ν denotes either I_ν or $\mathrm{e}^{\mathrm{i}\pi\nu} K_\nu$,

$$Z_{\nu-1}(z) - Z_{\nu+1}(z) = \frac{2\nu}{z} Z_\nu(z) \quad ; \quad Z_{\nu-1}(z) + Z_{\nu+1}(z) = 2\frac{\mathrm{d}Z_\nu(z)}{\mathrm{d}z}; \tag{C.65}$$

for $\nu = 0$ these imply

$$I_0'(z) = I_1(z) \quad ; \quad K_0'(z) = -K_1(z). \tag{C.66}$$

We shall use the results

$$e^{z\cos\theta} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\cos n\theta = \sum_{n=-\infty}^{\infty} I_n(z)\cos n\theta; \tag{C.67}$$

$$\int_0^\infty dx\, x^\mu J_\nu(x) = 2^\mu \frac{[\frac{1}{2}(\nu+\mu+3)]!}{[\frac{1}{2}(\nu-\mu+3)]!} \quad \mathrm{Re}(\nu+\mu) > -1,\ \mathrm{Re}(\mu) < \tfrac{1}{2}. \tag{C.68}$$

$$\begin{aligned}\int_u^\infty dx\, &\frac{x\,e^{-\mu x}}{(x^2-u^2)^{1-\nu}} \\ &= \frac{2^{\nu-1/2}(\nu-1)!}{\sqrt{\pi}}\frac{u^{\nu+1/2}}{\mu^{\nu-1/2}}K_{\nu+1/2}(u\mu) \qquad (u>0,\ \mathrm{Re}\,\mu,\nu>0);\end{aligned} \tag{C.69}$$

$$\begin{aligned}\int_0^b dy\, y^\nu (b^2-y^2)^{\nu-3/2} K_\nu(y) &= 2^{\nu-3}\sqrt{\pi}(\nu-\tfrac{3}{2})!\, b^{2\nu-1} \\ \times \big[I_{\nu-1}(\tfrac{1}{2}b)K_\nu(\tfrac{1}{2}b) - I_\nu(\tfrac{1}{2}b)K_{\nu-1}(\tfrac{1}{2}b)\big] &\qquad (\mathrm{Re}\,\nu > -\tfrac{1}{2}).\end{aligned} \tag{C.70}$$

Appendix D: Mechanics

We assume a background in classical mechanics at the advanced undergraduate level, including basic Hamiltonian mechanics. Useful reference texts include Landau & Lifshitz (1989), José & Saletan (1998), and Sussman & Wisdom (2001). The most elegant and mathematical treatment of the subject is found in Arnold (1989). This appendix contains a brief summary of the concepts employed in this book.

D.1 Single particles

The momentum of a particle is $\mathbf{p} = m\mathbf{v}$, where m is its mass and $\mathbf{v}$ is its velocity. Its motion is described by **Newton's second law**,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{D.1}$$

where $\mathbf{F}$ is the force acting on the particle. Thus, if the mass of the particle is constant,

$$\frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2} = \frac{\mathbf{F}}{m}. \tag{D.2}$$

The **work** done against the force $\mathbf{F}$ in moving a particle from $\mathbf{x}_1$ to $\mathbf{x}_2$ is

$$W_{12} = -\int_{\mathbf{x}_1}^{\mathbf{x}_2} d\mathbf{x}\cdot\mathbf{F}, \tag{D.3}$$

a line integral that is to be taken along the particle's trajectory from $\mathbf{x}_1$ to $\mathbf{x}_2$. The rate at which work is done on the force is

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathbf{x}_1}^{\mathbf{x}_2(t)} d\mathbf{x}\cdot\mathbf{F} = -\frac{d\mathbf{x}}{dt}\cdot\mathbf{F} = -\mathbf{F}\cdot\mathbf{v}, \tag{D.4}$$

evaluated at $\mathbf{x}_2$. For a particle of fixed mass,

$$\begin{aligned}W_{12} &= -m\int_{\mathbf{x}_1}^{\mathbf{x}_2} d\mathbf{x}\cdot\frac{d^2\mathbf{x}}{dt^2} = -m\int_{\mathbf{x}_1}^{\mathbf{x}_2} dt\,\frac{d\mathbf{x}}{dt}\cdot\frac{d^2\mathbf{x}}{dt^2} = -m\int_{\mathbf{x}_1}^{\mathbf{x}_2} dt\,\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\ &= \tfrac{1}{2}m[v^2(\mathbf{x}_1) - v^2(\mathbf{x}_2)].\end{aligned} \tag{D.5}$$