Homework 4 – AS.171.627 – Zakamska
AKA "MHD madness" – title courtesy of Duncan W.

1. Proper motion errors (2 points). Suppose that $N \gg 1$ measurements of the position of a star relative to several quasars are made at equal intervals Δt, all with the same uncertainties. The total timespan or the baseline of the observations is therefore $T = N\Delta t$. Assuming that the trigonometric parallax is negligible, how does the accuracy of the resulting proper motion determination scale with the length of the baseline? In other words, if the accuracy of the proper motion is $\epsilon \propto T^{-a}$, what is a?

2. Solar rotation speed (4 points). (a) Harris's catalog of Galactic globular clusters we used in a previous homework gives the Galactic latitude and longitude of each cluster (in Part I) as well as its radial velocity relative to the Local Standard of Rest (in Part III). Use these data to make a kinematic estimate of the rotation speed of the LSR, assuming that the cluster system itself does not rotate. Your result should include error bars. This is a kinematic estimate, not a dynamical estimate, i.e., you do not need to use Newton's laws. Does your result agree with the recent estimates? If not, what do you think may be the problem with this method? (b) Using the data in Part II of Harris's table, estimate the distance to M31, assuming that the luminosity function of globular clusters is the same in the two galaxies and that the mean apparent magnitude of the M31 globular clusters is $\langle m_V \rangle = 17.1$.

3. Ideal Magnetohydrodynamics (5+1+1+3 points). The equations of non-relativistic ideal MHD can be found in many places (e.g., on Wikipedia). Unfortunately, many places give them in SI rather than Gaussian (cgs) units. For compatibility with astrophysics literature I highly recommend the latter, so you need to be careful where you look. (You will also discover to your great dismay that some sources redefine the magnetic field to get rid of the 4π terms, $B_{\rm new} = B_{\rm Gauss}/\sqrt{4\pi}$, which is yet a third set of units, so watch out for this.) Ideal MHD describes the behavior of fluids in magnetic fields, with a great simplification that the fluid is assumed to be infinitely conductive. This is an excellent approximation for many astrophysical plasmas, for example quasar accretion disks. Every fluid element consists of electrons and positively charged nuclei, so its average charge is neutral, but if you apply an electric field to such fluid element, electrons and nuclei immediately start flowing in opposite directions with essentially no resistance and screen the field out.

Let's write out the equations of non-relativistic ideal MHD in Gaussian (cgs) units. We start with the continuity equation, and it is the same as the one we discussed in class:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0. \tag{1}$$

Euler equation acquires an Amper force term which should be familiar to you from basic E&M (just remember that it is per unit volume):

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right)\mathbf{v} = \frac{\mathbf{j}\times\mathbf{B}}{c} - \nabla p - \rho\nabla\Phi. \tag{2}$$

Here $\mathbf{j}$ is the current density (the standard current flowing through the wires that you used in basic E&M is I = current density times the cross-section of the wire). We have neglected the electrostatic force here because we've assumed that the plasma or the fluid has 0 net charge density. In addition, to describe the E&M fields we have three of the four Maxwell's equations which also should be familiar:

$$\nabla \mathbf{B} = 0 \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{5}$$

It turns out that in the non-relativistic case the *displacement current* term can be neglected. If you don't remember what this is, look it up and cross it off; we may discuss why this term is negligible at some later point, but for now let's just get rid of it.

The last Maxwell's equation connects the electric field with the charge density, which would then need its own equations which would be a big mess. Fortunately, under the assumptions of ideal MHD the electric field is screened because of the infinite conductivity in the frame co-moving with the fluid:

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = 0. \tag{6}$$

This equation replaces the 4th Maxwell's equation.

The final equation that wraps the system is the equation of state:

$$P = P(\rho). \tag{7}$$

These are the equations that are solved numerically when people study the behavior of non-relativistic plasmas in magnetic fields.

Let's consider a steady-state solution which is homogeneous fluid ρ_0 at rest $\mathbf{v_0} = 0$ with no gravity $\Phi = 0$ and uniform magnetic field $\mathbf{B_0}$ threading the fluid. $\mathbf{B_0}$ is directed for example along the z-axis (although you should try to keep the vector notation as long as possible). In addition we will consider the following simplifications:

- We will consider a fluid that is *incompressible*: its density is always ρ_0 (so $\rho_1 = 0$ in perturbation analysis). This is for example a good approximation for liquid metals that are used in labs to study MHD experimentally: metals are highly conductive, nearly incompressible, and if the metal is liquid at room temperature it makes the experiment set up that much easier. For example, Princeton Plasma Physics Lab had an MHD turbulence experiment filled with liquid gallium.

- We will ignore the pressure terms in the Euler (or momentum) equation, so we will consider them subdominant to all other forces.

- We will drop the displacement current which is unimportant in non-relativistic motion.
- We will only consider the perturbations in which $\mathbf{B_1} \perp \mathbf{B_0}$.

(a) First, determine to your satisfaction that the steady-state solution is in fact a solution to all the equations. Now let's see if this solution is stable. Conduct the linear stability analysis of this solution to perturbations with $\mathbf{B_1} \perp \mathbf{B_0}$. Derive the dispersion relation for these perturbations, find their phase and group velocity. Which way are they propagating? Which way are the fluid elements moving? Are these waves longitudinal or transverse? (I.e., are the fluid elements moving parallel or perpendicular to the direction of the wave propagation?) Are they growing or decaying? What are they called? (Hint: there are no curvy derivatives anywhere in this problem! If you need to use coordinates, use the Cartesian ones.)

(b) Calculate the group velocity of these perturbations for the HII gas in the disk of the Milky Way. Some information on the phases of the interstellar medium is summarized in the intro to the book by B.Draine "Physics of the interstellar and intergalactic medium" (see attached, although you still may need to look up B_0). Compare this velocity to the sound speed in the same gas. Provide references for numerical values if necessary.

(c) Are ideal MHD equations appropriate for describing the structure and evolution of a protoplanetary disk? Why?

(d) Derive equation (6) from relativistic EM field transforms and from the condition that the electric field in the frame co-moving with infinitely conductive fluid must be zero. This equation is strongly related to the "flux freezing equation":

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{8}$$

Can you derive it from the equations provided above? Is it applicable to relativistic plasmas? Is it applicable to compressible plasmas? Why is this equation called "the flux freezing equation"? What is frozen to what?

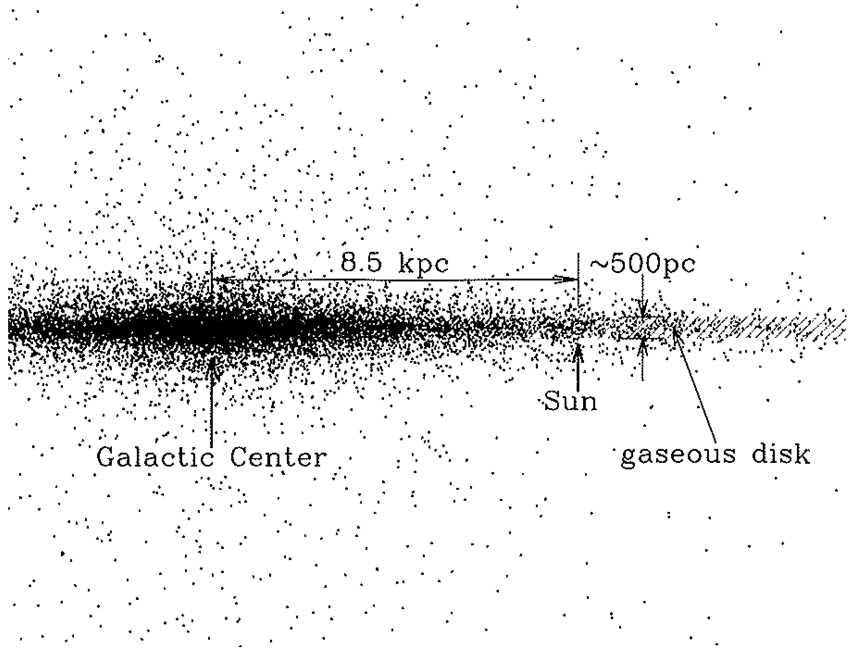

Figure 1.2 Structure of the Milky Way, viewed edge-on. The dots represent a sampling of stars; the volume containing most of the interstellar gas and dust is shaded. Compare with the infrared image of the stars in Plate 1, the dust in Plate 2, and various gas components in Plates 3–5.

1.1 Organization of the ISM: Characteristic Phases

In a spiral galaxy like the Milky Way, most of the dust and gas is to be found within a relatively thin gaseous disk, with a thickness of a few hundred pc (see the diagram in Fig. 1.2 and the images in Plates 1–5), and it is within this disk that nearly all of the star formation takes place. While the ISM extends above and below this disk, much of our attention will concern the behavior of the interstellar matter within a few hundred pc of the disk midplane.

The Sun is located about 8.5 kpc from the center of the Milky Way; as it happens, the Sun is at this time very close to the disk midplane. The total mass of the Milky Way within 15 kpc of the center is approximately $10^{11} M_\odot$; according to current estimates, this includes $\sim 5 \times 10^{10} M_\odot$ of stars, $\sim 5 \times 10^{10} M_\odot$ of dark matter, and $\sim 7 \times 10^9 M_\odot$ of interstellar gas, mostly hydrogen and helium (see Table 1.2). About 60% of the interstellar hydrogen is in the form of H atoms, ~20% is in the form of H_2 molecules, and ~20% is ionized.

The gaseous disk is approximately symmetric about the midplane, but does not have a sharp boundary – it is like an atmosphere. We can define the half-thickness

Table 1.2 Mass of H II, H I, and H_2 in the Milky Way ($R < 20$ kpc)

Phase	$M(10^9 M_\odot)$	fraction	Note
Total H II (not including He)	1.12	23%	see Chapter 11
Total H I (not including He)	2.9	60%	see Chapter 29
Total H_2 (not including He)	0.84	17%	see Chapter 32
Total H II, H I and H_2 (not including He)	4.9		
Total gas (including He)	6.7		

$z_{1/2}$ of the disk to be the distance z above (or below) the plane where the density has dropped to 50% of the midplane value. Observations of radio emission from atomic hydrogen and from the CO molecule indicate that the half-thickness $z_{1/2} \approx 250$ pc in the neighborhood of the Sun. The thickness $2z_{1/2} \approx 500$ pc of the disk is only ~6% of the ~8.5 kpc distance from the Sun to the Galactic center – it is a *thin* disk. The thinness of the distribution of dust and gas is evident from the 100 μm image showing thermal emission from dust in Plate 2, and the H I 21-cm line image in Plate 3.

The baryons in the interstellar medium of the Milky Way are found with a wide range of temperatures and densities; because the interstellar medium is dynamic, all densities and temperatures within these ranges can be found somewhere in the Milky Way. However, it is observed that most of the baryons have temperatures falling close to various characteristic states, or "phases." For purposes of discussion, it is convenient to name these phases. Here we identify seven distinct phases that, between them, account for most of the mass and most of the volume of the interstellar medium. These phases (summarized in Table 1.3) consist of the following:

- **Coronal gas**: Gas that has been shock-heated to temperatures $T \gtrsim 10^{5.5}$ K by blastwaves racing outward from supernova explosions. The gas is collisionally ionized, with ions such as O VI ($\equiv O^{5+}$) present. Most of the coronal gas has low density, filling an appreciable fraction – approximately half – of the volume of the galactic disk. The coronal gas regions may have characteristic dimensions of ~ 20 pc, and may be connected to other coronal gas volumes. The coronal gas cools on ~Myr time scales. Much of the volume above and below the disk is thought to be pervaded by coronal gas.[1] It is often referred to as the "hot ionized medium," or **HIM**.
- **H II gas**: Gas where the hydrogen has been photoionized by ultraviolet photons from hot stars. Most of this photoionized gas is maintained by radiation from recently formed hot massive O-type stars – the photoionized gas may be dense material from a nearby cloud (in which case the ionized gas is called an **H II region**) or lower density "intercloud" medium (referred to as **diffuse H II**).

[1] This gas is termed "coronal" because its temperature and ionization state is similar to the corona of the Sun.

Table 1.3 Phases of Interstellar Gas

Phase	T(K)	n_H(cm^{-3})	Comments
Coronal gas (HIM) $f_V \approx 0.5?$ $\langle n_H \rangle f_V \approx 0.002\,\mathrm{cm}^{-3}$ ($f_V \equiv$ volume filling factor)	$\gtrsim 10^{5.5}$	~ 0.004	Shock-heated Collisionally ionized Either expanding or in pressure equilibrium Cooling by: ◇ Adiabatic expansion ◇ X ray emission Observed by: • UV and x ray emission • Radio synchrotron emission
H II gas $f_V \approx 0.1$ $\langle n_H \rangle f_V \approx 0.02\,\mathrm{cm}^{-3}$	10^4	$0.3 - 10^4$	Heating by photoelectrons from H, He Photoionized Either expanding or in pressure equilibrium Cooling by: ◇ Optical line emission ◇ Free–free emission ◇ Fine-structure line emission Observed by: • Optical line emission • Thermal radio continuum
Warm H I (WNM) $f_V \approx 0.4$ $n_H f_V \approx 0.2\,\mathrm{cm}^{-3}$	~5000	0.6	Heating by photoelectrons from dust Ionization by starlight, cosmic rays Pressure equilibrium Cooling by: ◇ Optical line emission ◇ Fine structure line emission Observed by: • H I 21 cm emission, absorption • Optical, UV absorption lines
Cool H I (CNM) $f_V \approx 0.01$ $n_H f_V \approx 0.3\,\mathrm{cm}^{-3}$	~ 100	30	Heating by photoelectrons from dust Ionization by starlight, cosmic rays Cooling by: ◇ Fine structure line emission Observed by: • H I 21-cm emission, absorption • Optical, UV absorption lines
Diffuse H_2 $f_V \approx 0.001$ $n_H f_V \approx 0.1\,\mathrm{cm}^{-3}$	~ 50 K	~ 100	Heating by photoelectrons from dust Ionization by starlight, cosmic rays Cooling by: ◇ Fine structure line emission Observed by: • H I 21-cm emission, absorption • CO 2.6-mm emission • optical, UV absorption lines
Dense H_2 $f_V \approx 10^{-4}$ $\langle n_H \rangle f_V \approx 0.2\,\mathrm{cm}^{-3}$	$10 - 50$	$10^3 - 10^6$	Heating by photoelectrons from dust Ionization and heating by cosmic rays Self-gravitating: $p > p$(ambient ISM) Cooling by: ◇ CO line emission ◇ C I fine structure line emission Observed by: • CO 2.6-mm emission • dust FIR emission
Cool stellar outflows	$50 - 10^3$	$1 - 10^6$	Observed by: • Optical, UV absorption lines • Dust IR emission

Bright H II regions, such as the Orion Nebula, have dimensions of a few pc; their lifetimes are essentially those of the ionizing stars, $\sim 3 - 10$ Myr. The extended low-density photoionized regions – often referred to as the **warm ionized medium**, or **WIM** – contain much more total mass than the more visually conspicuous high-density H II regions. According to current estimates, the Galaxy contains $\sim 1.1 \times 10^9\,M_\odot$ of ionized hydrogen; about 50% of this is within 500 pc of the disk midplane (the distribution of the H II is discussed in Chapter 11). In addition to the H II regions, photoionized gas is also found in distinctive structures called **planetary nebulae**[2] – these are created when rapid mass loss during the late stages of evolution of stars with initial mass $0.8M_\odot < M < 6M_\odot$ exposes the hot stellar core; the radiation from this core photoionizes the outflowing gas, creating a luminous (and often very beautiful) planetary nebula. Individual planetary nebulae fade away on $\sim 10^4$ yr time scales.

- **Warm H I**: Predominantly atomic gas heated to temperatures $T \approx 10^{3.7}$ K; in the local interstellar medium, this gas is found at densities $n_H \approx 0.6\,\mathrm{cm}^{-3}$. It fills a significant fraction of the volume of the disk – perhaps 40%. Often referred to as the **warm neutral medium**, or **WNM**.

- **Cool H I**: Predominantly atomic gas at temperatures $T \approx 10^2$ K, with densities $n_H \approx 30\,\mathrm{cm}^{-3}$ filling ~1% of the volume of the local interstellar medium. Often referred to as the **cold neutral medium**, or **CNM**.

- **Diffuse molecular gas**: Similar to the cool H I clouds, but with sufficiently large densities and column densities so that H_2 self-shielding (discussed in Chapter 31) allows H_2 molecules to be abundant in the cloud interior.

- **Dense molecular gas**: Gravitationally bound clouds that have achieved $n_H \gtrsim 10^3\,\mathrm{cm}^{-3}$. These clouds are often "dark" – with visual extinction $A_V \gtrsim 3$ mag through their central regions. In these dark clouds, the dust grains are often coated with "mantles" composed of H_2O and other molecular ices. It is within these regions that star formation takes place. It should be noted that the gas pressures in these "dense" clouds would qualify as ultrahigh vacuum in a terrestrial laboratory.

- **Stellar outflows**: Evolved cool stars can have mass loss rates as high as $10^{-4} M_\odot\,\mathrm{yr}^{-1}$ and low outflow velocities $\lesssim 30\,\mathrm{km\,s}^{-1}$, leading to relatively high density outflows. Hot stars can have winds that are much faster, although far less dense.

The ISM is dynamic, and the baryons undergo changes of phase for a number of reasons: ionizing photons from stars can convert cold molecular gas to hot H II; radiative cooling can allow hot gas to cool to low temperatures; ions and electrons can recombine to form atoms, and H atoms can recombine to form H_2 molecules.

[2]They are called "planetary" nebulae because of their visual resemblance to planets when viewed through a small telescope.