where $\varepsilon(\omega) = \varepsilon_0(1 + \chi(\omega)) + i\sigma(\omega)/\omega$. Note that the two sets of equations (2.12)–(2.15) and (2.28)–(2.31) are formally identical although interpreted differently. In the former set, $(\mathbf{E}_c, \mathbf{H}_c)$ is the complex representation of a time-harmonic field; whereas in the latter set, $(\mathcal{E}, \mathcal{H})$ is the Fourier transform of an arbitrarily time-dependent electromagnetic field. But the same equations have the same solutions, so we need consider only time-harmonic fields; the general time-dependent field can be constructed by Fourier synthesis.

Both the conductivity and the susceptibility contribute to the imaginary part of the permittivity: $\text{Im}\{\varepsilon\} = \varepsilon_0 \,\text{Im}\{\chi\} + \text{Re}\{\sigma/\omega\}$. A nonzero value for $\text{Im}\{\varepsilon\}$ manifests itself physically by absorption of electromagnetic energy in the medium. We may associate $\text{Im}(\chi)$ with the "bound" charge current density and $\text{Re}(\sigma/\omega)$ with the "free" charge current density. Absorption is determined by the *sum* of these two quantities, however, and it is not possible to determine by absorption measurements their relative contributions. This underscores our assertion that there is no clearly defined distinction between "free" and "bound" charges.

2.3.2 Kramers–Kronig Relations

As an alternative we could have begun our discussion of constitutive relations with the *assumption* that $\mathbf{P}(t)$ and $\mathbf{E}(t)$ are related through the linear functional equation

$$\mathbf{P}(t) = \int_{-\infty}^{\infty} G(t, t')\mathbf{E}(t')\,dt'. \tag{2.32}$$

Suppose that the electric field is a delta function applied at time t_0: $\mathbf{E}(t) = \delta(t - t_0)\mathbf{E}_0$; the corresponding polarization is therefore

$$\mathbf{P}(t) = G(t, t_0)\mathbf{E}_0.$$

Thus, G is the polarization resulting from a unit amplitude delta function. If the properties of the medium do not change with time, the polarization must depend only on the time elapsed between t_0 and t:

$$G(t, t_0) = G(t - t_0).$$

Therefore, we obtain (2.26), which when inverted yields the constitutive relation (2.23). We note that *causality*—the system cannot squeal before it is hurt—requires that $G(\tau) = 0$ for $\tau < 0$.

The susceptibility is the Fourier transform of $G(t)$:

$$\varepsilon_0\chi(\omega) = \int_{-\infty}^{\infty} G(t)e^{i\omega t}\,dt = \int_0^{\infty} G(t)e^{i\omega t}\,dt, \tag{2.33}$$

and is a complex-valued function of the real variable ω. Let us *define* a complex-valued function of the *complex* variable $\tilde{\omega}$ by

$$\epsilon_0\chi(\tilde{\omega}) = \int_0^\infty G(t)e^{i\tilde{\omega}t}\,dt,$$

where $\tilde{\omega} = \omega_r + i\omega_i$. The function $\chi(\tilde{\omega})$ coincides with $\chi(\omega)$ when $\tilde{\omega}$ is a point on the real axis. For any $t \geqslant 0$, $G(t)e^{i\tilde{\omega}t}$ is an analytic function of $\tilde{\omega}$, and $|G(t)e^{i\tilde{\omega}t}| \leqslant |G(t)|$ if $\omega_i > 0$. Therefore, if the integral

$$\int_0^\infty |G(t)|dt \tag{2.34}$$

converges, then

$$\int_0^\infty G(t)e^{i\tilde{\omega}t}\,dt$$

converges to an analytic function in the upper half of the complex $\tilde{\omega}$ plane. Convergence of (2.34) is assured if $\chi(0)$ is finite; this follows from (2.33). If $\chi(\tilde{\omega})$ is analytic, so is $\chi(\tilde{\omega})/(\tilde{\omega} - \omega)$ except at the pole $\tilde{\omega} = \omega$. Cauchy's theorem states that

$$\int_C f(\tilde{\omega})\,d\tilde{\omega} = 0,$$

provided that the closed contour C encloses no poles of the analytic function $f(\tilde{\omega})$. Let us apply Cauchy's theorem to the function $\chi(\tilde{\omega})/(\tilde{\omega} - \omega)$, where ω is a point on the real axis, and the contour C, shown in Fig. 2.1, is the union of four curves with parametric representations

$$C_1:\quad \tilde{\omega} = \Omega \qquad (-A \leqslant \Omega \leqslant \omega - a),$$

$$C_2:\quad \tilde{\omega} = \omega - ae^{-i\Omega} \qquad (0 \leqslant \Omega \leqslant \pi),$$

$$C_3:\quad \tilde{\omega} = \Omega \qquad (\omega + a \leqslant \Omega \leqslant A),$$

$$C_4:\quad \tilde{\omega} = Ae^{i\Omega} \qquad (0 \leqslant \Omega \leqslant \pi).$$

Therefore, from Cauchy's theorem

$$\int_{-A}^{\omega-a} \frac{\chi(\Omega)}{\Omega - \omega}\,d\Omega + \int_{\omega+a}^{A} \frac{\chi(\Omega)}{\Omega - \omega}\,d\Omega + \int_0^\pi \frac{iAe^{i\Omega}\chi(Ae^{i\Omega})}{Ae^{i\Omega} - \omega}\,d\Omega = \int_0^\pi i\chi(\omega - ae^{i\Omega})\,d\Omega.$$

The integral over the curve C_4 vanishes as A tends to infinity if $\lim_{|\tilde{\omega}|\to\infty}\chi(\tilde{\omega})$

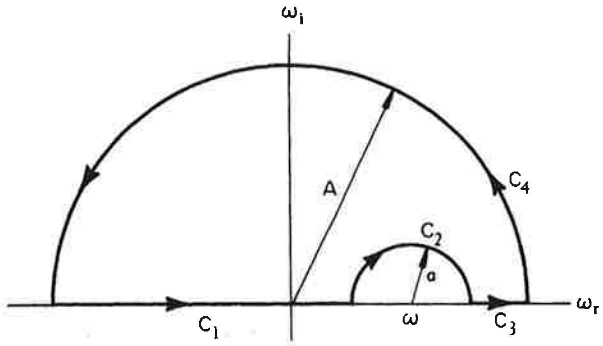

Figure 2.1 Contour of integration.

= 0; this implies that

$$i\pi\chi(\omega) = P\int_{-\infty}^{\infty} \frac{\chi(\Omega)}{\Omega - \omega} d\Omega, \tag{2.35}$$

where the symbol P denotes the *Cauchy principal value* of the integral, defined by

$$\lim_{a\to 0} \int_{-\infty}^{\omega - a} \frac{\chi(\Omega)}{\Omega - \omega} d\Omega + \int_{\omega + a}^{\infty} \frac{\chi(\Omega)}{\Omega - \omega} d\Omega.$$

We were slightly careless in our derivation of (2.35): $\chi(\tilde{\omega})$ was required to be analytic in the upper half of the complex $\tilde{\omega}$ plane, whereas part of the contour C included the real axis. But this can easily be remedied, and a more general form of (2.35) is

$$i\pi \lim_{\eta\to 0} \chi(\omega + i\eta) = \lim_{\eta\to 0} P\int_{-\infty}^{\infty} \frac{\chi(\Omega + i\eta)}{\Omega - \omega} d\Omega,$$

from which (2.35) follows if $\chi(\tilde{\omega})$ is continuous.

The fundamental relation (2.35) can be written as two real integral relations:

$$\chi'(\omega) = \frac{2}{\pi} P\int_0^{\infty} \frac{\Omega\chi''(\Omega)}{\Omega^2 - \omega^2} d\Omega, \tag{2.36}$$

$$\chi''(\omega) = \frac{-2\omega}{\pi} P\int_0^{\infty} \frac{\chi'(\Omega)}{\Omega^2 - \omega^2} d\Omega, \tag{2.37}$$

where $\chi = \chi' + i\chi''$ and we have invoked the *crossing condition* $\chi^*(\omega) = \chi(-\omega)$. Equations (2.36) and (2.37) are an important example of a rather remarkable class of mathematical relations called *Kramers–Kronig* or *dispersion relations*. Their implication, which is unexpected on physical grounds and therefore has an aura of black magic, is that the real and imaginary parts of χ are not independent but are connected by integral relations; this imposes a constraint on physically realizable susceptibilities. Moreover, if χ' is known over a sufficiently large range of frequencies around ω, $\chi''(\omega)$ can be obtained by integration, and vice versa. An interesting corollary of (2.36) is the *sum rule*:

$$\chi'(0) = \frac{2}{\pi}\int_0^\infty \frac{\chi''(\Omega)}{\Omega}\,d\Omega.$$

Although the Kramers–Kronig relations do not follow directly from physical reasoning, they are not devoid of physical content: underlying their derivation are the assumptions of *linearity* and *causality* and restrictions on the asymptotic behavior of χ. As we shall see in Chapter 9, the required asymptotic behavior of χ is a physical consequence of the interaction of a frequency-dependent electric field with matter.

The derivation of Kramers–Kronig relations for the susceptibility was relatively easy, perhaps misleadingly so. With a bit of extra effort, however, we can often derive similar relations for other frequency-dependent quantities that arise in physical problems. Suppose that we have two time-dependent quantities of unspecified origin, which we may call the *input* $X_i(t)$ and the *output* $X_o(t)$; the corresponding Fourier transforms are denoted by $\mathcal{X}_i(\omega)$ and $\mathcal{X}_o(\omega)$. If the relation between these transforms is linear,

$$\mathcal{X}_o(\omega) = \mathcal{R}(\omega)\mathcal{X}_i(\omega),$$

and causal (i.e., the output cannot precede the input in time), then $\mathcal{R}(\tilde{\omega})$ is an analytic function in the top half of the complex $\tilde{\omega}$ plane. It is also necessary that $\mathcal{R}(\tilde{\omega})$ vanish on the circular arc C_4 (Fig. 2.1) as A approaches infinity; if it does not, we are permitted to fiddle with it until it does. That is, we can change the asymptotic behavior of $\mathcal{R}(\tilde{\omega})$ by multiplying it by some analytic function $g(\tilde{\omega})$, or adding $g(\tilde{\omega})$ to it, without changing its analyticity. Of course, in so doing, we may also change the crossing condition, and the resultant dispersion relations may be different from (2.36) and (2.37). Techniques of fiddling with $\mathcal{R}$ until it behaves properly are best illustrated with specific examples, which we shall encounter later in this chapter and elsewhere in the book.

2.4 SPATIAL DISPERSION

We have shown that a frequency-dependent susceptibility implies *temporal dispersion*: the polarization at time t depends on the electric field at all times previous to t. It is also possible under some circumstances to have *spatial dispersion*: the polarization at point $\mathbf{x}$ depends on the values of the electric field at points in some neighborhood of $\mathbf{x}$. This nonlocal relation between $\mathbf{P}$ and $\mathbf{E}$