so that the Lorentz condition is satisfied; the primes in the potentials can then be omitted.

The equations for potentials. From the Lorentz condition and (12.33), we obtain the equation for a vector potential:

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi \mathbf{j}}{c}. \qquad (12.37)$$

It is now also easy to obtain the equation for a scalar potential. From (12.27) we have

$$\operatorname{div} \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{div} \mathbf{A} - \Delta\varphi = 4\pi\rho.$$

Substituting div **A** from the Lorentz condition (12.36), we obtain

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho. \qquad (12.38)$$

Equations (12.37) and (12.38) each contain only one unknown. Therefore, each equation for potential does not depend on the rest and can be solved separately.

The equations for potential are second order with respect to coordinate and time derivatives. For a solution, it is necessary to give not only the initial values of the potentials, but also the initial values of their time derivatives.

Gauge invariance. As we shall see later, especially in the following section and in Sec. 21, which is devoted to the motion of charges in an electromagnetic field, it is necessary, in many cases, to use equations involving potentials. But, since potentials are ambiguous, we must take care that the form of any equation involving potentials does not change under gauge transformations (12.30) and (12.31),* since such transformations involve a completely arbitrary function f which can be chosen to be of any form. It is clear that no physical result can depend on the choice of this function, i.e., on an arbitrary gauge transformation. In other words, equations involving potentials must be *gauge invariant*.

Sec. 13. The Action Principle for the Electromagnetic Field

The variational principle for the electromagnetic field. In the first part of this book it was shown that the equations of mechanics, obtained from Newton's laws (Sec. 2), lead to the principle of least action (Sec. 10). We obtained the equations of electrodynamics in the preceding section by proceeding from certain simple physical laws and the assumption about the magnetic effect due to displacement current. In this section, Maxwell's equations will be reduced to the variational principle, which is the principle of least action for the electromagnetic field.

* This does not refer to equations (12.37) and (12.38), from which the potentials are determined in accordance with the condition (12.36).

Electrodynamics is not equivalent to the mechanics of particle systems or to the mechanics of liquids, which are based on Newton's laws. All the same, to a very considerable extent, electrodynamical laws are analogous to the laws of mechanics. This analogy can best be seen from the principle of least action for the electromagnetic field.

The variational formulation best of all allows us to derive the conservation laws for the electromagnetic field. The corresponding integrals of motion for a field coincide with the well-known mechanical integrals—energy, linear momentum, and angular momentum. In a closed system consisting of charged particles and a field, the total energy, total linear momentum, and total angular momentum of the charges and field are conserved.

In this sense, electrodynamics is indeed "a dynamics" of the electromagnetic field, though this by no means signifies that the laws of electrodynamics can be obtained from Newton's laws. Both are equivalent to certain integral variational principles, but the action functions are, of course, of entirely different form.

It is a noteworthy fact that Maxwell at first tried to construct mechanical models of the ether, but in his later work he rejected them and obtained the general equations of electrodynamics by means of a generalization of known elementary laws of electromagnetism.

The Lagrangian function for a field. In order to formulate the principle of least action it is necessary to have an expression for the Lagrangian. The choice of Lagrangian in mechanics is determined by considerations based on the relativity principle of Newtonian mechanics, which is formulated with the aid of Galilean transformations (Sec. 8). As will be explained in detail in Secs. 20 and 21, Galilean transformations are not valid in electrodynamics and are replaced by the more general Lorentz transformations, based on the Einstein relativity principle. These transformations allow the Lagrangian for the electromagnetic field to be uniquely found; this will be done in Sec. 21. In this section, the choice of Lagrangian is justified by the fact that the already familiar Maxwell equations are obtained from it. Similarly, in Part I, the principle of least action was formulated after Lagrange's equations had been obtained on the basis of Newton's laws. This confirmed the truth of the integral principle.

In finding the Lagrangian for a system of free particles, a summation is performed over the coordinates of the particles. The electromagnetic field, if we use the terminology of mechanics, is a system with an infinite number of degrees of freedom because, for a complete description of the field, we must know all its components at all points of space, where they differ from zero. But the points of space form a nondenumerable set, i.e., they cannot be numbered in any order. For this reason, for the electromagnetic field the summation in the Lagrangian is replaced by an integration with respect to continuously varying parameters, i.e., coordinates of points in which the field is given. The point coordinates are analogous to the indices which label the degrees of freedom of a mechanical system.

The equations of mechanics are second order in time with respect to generalized coordinates q_k. The equations for potentials (12.37) and (12.38) are also second order in time. Therefore, potential quantities should be chosen as the generalized coordinates.

In other words, $\mathbf{A}(\mathbf{r}, t)$, $\varphi(\mathbf{r}, t)$ correspond to $q_k(t)$, where $\mathbf{A}$ and φ are potentials which are generalized coordinates of an electromagnetic field. The value of the radius vector $\mathbf{r}$ for the point at which the potential is taken corresponds to the number of the generalized coordinate k.

In order to write down the complete Lagrangian function, we must first of all define it in an element of volume dV and integrate over the volume occupied by the field. It has already been mentioned that in this section we will proceed immediately from a Lagrangian that leads to correct Maxwell's equations; the choice of this Lagrangian as based on considerations related to the relativity principle will be left to Sec. 21. The Lagrangian is of the following form:

$$L=\int\left(\frac{\mathbf{E}^2-\mathbf{H}^2}{8\pi}+\frac{\mathbf{A}\mathbf{j}}{c}-\rho\varphi\right)dV. \tag{13.1}$$

Since the potentials are liable to be generalized coordinates of the field, expression (13.1) should be rewritten thus:

$$L=\int\left[\frac{1}{8\pi}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)^2-\frac{1}{8\pi}(\operatorname{rot}\mathbf{A})^2+\frac{\mathbf{A}\mathbf{j}}{c}-\rho\varphi\right]dV. \tag{13.2}$$

Here, in place of a summation over the degrees of freedom, an integration over the volume has been performed.

The extremal property of action in electrodynamics. We shall now show that action, i.e., $S=\int L\,dt$, possesses the same variational property in electrodynamics as it does in mechanics: its variation becomes zero if the field satisfies the correct equations of motion (in this case, the Maxwell equations).

We shall begin with variation with respect to the scalar potential φ:

$$\delta_\varphi L=\int\left[\frac{1}{4\pi}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)^2\delta\nabla\varphi-\rho\delta\varphi\right]dV. \tag{13.3}$$

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As was shown in Sec. 10, variation and differentiation are commutative so that $\delta\nabla\varphi=\nabla\delta\varphi$. According to (12.29) we replace the term $\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)$ by $-\mathbf{E}$. Therefore,

$$\delta_\varphi L=\int\left(-\frac{\mathbf{E}\nabla\delta\varphi}{4\pi}-\rho\delta\varphi\right)dV. \tag{13.4}$$

We shall now make use of equation (11.27), in accordance with which

$$\mathbf{E}\nabla\delta\varphi=\operatorname{div}(\mathbf{E}\delta\varphi)-\delta\varphi\operatorname{div}\mathbf{E}. \tag{13.5}$$

We then obtain

$$\delta_\varphi L=\int\left[\frac{-\operatorname{div}(\delta\varphi\mathbf{E})}{4\pi}+\delta\varphi\left(\frac{\operatorname{div}\mathbf{E}}{4\pi}-\rho\right)\right]dV. \tag{13.6}$$

The first term in (13.6) can be transformed into a surface integral, so that $\delta_\varphi L$ will have the form

$$\delta_\varphi L=-\int\delta\varphi\,\mathbf{E}\,d\mathbf{s}+\int\delta\varphi\left(\frac{\operatorname{div}\mathbf{E}}{4\pi}-\rho\right)dV. \tag{13.7}$$

We shall consider that the first integral is taken over a surface on which $\delta\varphi$ becomes zero, similar to the way that, in Sec. 10, δq was equal to zero at the limits of integration (a surface is the limit for a volume integral).

Therefore,

$$\delta_\varphi L=\int\delta\varphi\left(\frac{\operatorname{div}\mathbf{E}}{4\pi}-\rho\right)dV. \tag{13.8}$$

However, since

$$\operatorname{div}\mathbf{E}=4\pi\rho \tag{13.9}$$

[see (12.27)], $\delta_\varphi L$ and hence $\delta_\varphi S$, becomes zero as expected.

Let us now vary L with respect to $\mathbf{A}$. This variation has the form

$$\delta_{\mathbf{A}}L=\int\left[\frac{1}{4\pi}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)\frac{1}{c}\delta\frac{\partial\mathbf{A}}{\partial t}-\frac{1}{4\pi}\operatorname{rot}\mathbf{A}\,\delta\operatorname{rot}\mathbf{A}+\frac{\mathbf{j}\delta\mathbf{A}}{c}\right]dV. \tag{13.10}$$

Once again we interchange the differentiation and variation signs and, where possible, we replace the potentials by fields after variation. We obtain $\delta_A L$ in the following form:

$$\delta_{\mathbf{A}}L=\int\left[-\frac{1}{4\pi c}\mathbf{E}\frac{\partial}{\partial t}\delta\mathbf{A}-\frac{1}{4\pi}\mathbf{H}\operatorname{rot}\delta\mathbf{A}+\frac{\mathbf{j}\,\delta\mathbf{A}}{c}\right]dV. \tag{13.11}$$

Let us write down the transformation by parts:

$$\mathbf{E}\frac{\partial}{\partial t}\delta\mathbf{A}=\frac{\partial}{\partial t}\mathbf{E}\delta\mathbf{A}-\delta\mathbf{A}\frac{\partial\mathbf{E}}{\partial t}, \tag{13.12}$$

$$\mathbf{H}\operatorname{rot}\delta\mathbf{A}=-\operatorname{div}[\mathbf{H}\delta\mathbf{A}]+\delta\mathbf{A}\operatorname{rot}\mathbf{H}. \tag{13.13}$$

The last equation follows from (11.29). To take advantage of (13.12), we must write down the variation of the action S instead of the variation of L. Then the first term of (13.12) can be directly integrated with respect to time, and $\delta_A S$ will be

$$\delta_A S=\int_{t_0}^{t_1}\delta_A L dt=-\frac{1}{4\pi c}\int \mathbf{E}\delta\mathbf{A}\,dV\Big|_{t_0}^{t_1}+\frac{1}{4\pi}\int_{t_0}^{t_1}dt\int[\mathbf{H}\delta\mathbf{A}]\,ds+$$
$$+\int_{t_0}^{t_1}dt\int dV\left(-\frac{\operatorname{rot}\mathbf{H}}{4\pi}+\frac{1}{4\pi c}\frac{\partial\mathbf{E}}{\partial t}+\frac{\mathbf{j}}{c}\right)\delta\mathbf{A}. \quad (13.14)$$

The variation $\delta\mathbf{A}$ is equal to zero at the initial and final instants of time t_0 and t_1, as well as over the surface bounding the field. Therefore,

$$\delta_A S=\int_{t_0}^{t_1}dt\int dV\left(-\frac{\operatorname{rot}\mathbf{H}}{4\pi}+\frac{1}{4\pi c}\frac{\partial\mathbf{E}}{\partial t}+\frac{\mathbf{j}}{c}\right)\delta\mathbf{A}, \quad (13.15)$$

and since the field satisfies the equation

$$\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi\mathbf{j}}{c} \quad (13.16)$$

[see (12.26)], $\delta_A S$ is equal to zero.

The first pair of Maxwell's equations is, of course, satisfied identically if the fields are expressed in terms of potentials in accordance with (12.28) and (12.29).

Thus, Maxwell's equations can be interpreted as equations of the mechanics of an electromagnetic field. They could be obtained from the variational method, starting from the Lagrangian (13.1) and the requirement that the variation of action should be equal to zero for any arbitrary variations of the scalar and vector potentials. For this it is sufficient to repeat the arguments set out in Sec. 10, as applied to integrals (13.8) and (13.16).

The invariance of action with respect to a potential gauge transformation. We shall now show that action is invariant under gauge transformations (12.30) and (12.31), despite the fact that it involves not only fields, but also potentials contained in the last two terms of equation (13.1). We shall call the corresponding part of the action S_1:

$$S_1=\int dt\int dV\left(\frac{\mathbf{A}\mathbf{j}}{c}-\rho\varphi\right). \quad (13.17)$$

Let us now apply gauge transformations (12.30) and (12.31) to $\mathbf{A}$ and φ. This gives

$$S_1=\int dt\int dV\left(\frac{\mathbf{A}'\mathbf{j}}{c}-\rho\varphi'+\frac{\mathbf{j}\nabla f}{c}+\rho\frac{1}{c}\frac{\partial f}{\partial t}\right). \quad (13.18)$$

We transform by parts terms containing f:

$$\mathbf{j}\nabla f=\operatorname{div}(f\mathbf{j})-f\operatorname{div}\mathbf{j},\quad \rho\frac{\partial f}{\partial t}=\frac{\partial}{\partial t}(\rho f)-f\frac{\partial\rho}{\partial t}.$$

Substituting this in the integral S_1 and performing the integration, as in (13.14), we have

$$S_1=\frac{1}{c}\int dt\int ds\,f\mathbf{j}-\frac{1}{c}\int dV\,f\rho\Big|_{t_0}^{t_1}+$$
$$+\int dt\int dV\left[\frac{\mathbf{A}'\mathbf{j}}{c}-\rho\varphi'-f\left(\operatorname{div}\mathbf{j}+\frac{\partial\rho}{\partial t}\right)\right]. \quad (13.19)$$

However, the integrated terms do not affect the Maxwell equations since, when performing a variation of S_1, both $\delta_\varphi L$ and $\delta_A L$ are equal to zero at the boundaries of the integration region. We have already encountered this in (10.9). The term, proportional to f, under the integral sign, is multiplied by the quantity $\operatorname{div}\mathbf{j}+\frac{\partial\rho}{\partial t}$, which is identically equal to zero according to the charge-conservation law (12.18). Thus, S_1 retains the form (13.17).

The energy of a field. Maxwell's equations also apply to a free electromagnetic field not containing charges or currents. For this it is sufficient to omit from them the terms $\frac{4\pi\mathbf{j}}{c}$ and $4\pi\rho$. In accordance with (13.1) and (13.2), the Lagrangian for a free field is

$$L_0=\int\frac{\mathbf{E}^2-\mathbf{H}^2}{8\pi}dV=\int\frac{1}{8\pi}\left[\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)^2-(\operatorname{rot}\mathbf{A})^2\right]dV. \quad (13.20)$$

We shall now determine the energy of an electromagnetic field by proceeding from the general equation (4.4). First, let it be recalled that the values of potentials at all points of space are generalized coordinates. But then the derivatives $\frac{\partial\mathbf{A}}{\partial t}$ are generalized velocities. Consequently, the expression

$$\mathscr{E}=\sum_k\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-L$$

reduces to the form

$$\mathscr{E}=\int dV\left[\frac{1}{4\pi c}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right)\frac{\partial\mathbf{A}}{\partial t}-\left(\frac{\mathbf{E}^2-\mathbf{H}^2}{8\pi}\right)\right]$$

by means of the comparison

$$\dot{q}_k\sim\frac{\partial\mathbf{A}}{\partial t},\quad \frac{\partial L}{\partial\dot{q}_k}\sim\frac{1}{4\pi c}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi\right),\ \sum_k\sim\int dV.$$

We shall now show that energy is expressed only in terms of the field, and not in terms of potentials. Using (12.29) we write down the energy thus:

$$\mathscr{E}=\int dV\left[\frac{1}{4\pi}\mathbf{E}(\mathbf{E}+\nabla\varphi)-\left(\frac{\mathbf{E}^2-\mathbf{H}^2}{8\pi}\right)\right].$$

This expression is not invariant with respect to a gauge transformation and must be transformed.

Transforming the term $\mathbf{E}\nabla\varphi$ by parts, we have, from (11.27),

$$\mathbf{E}\nabla\varphi=\operatorname{div}(\mathbf{E}\varphi)-\varphi\operatorname{div}\mathbf{E}=\operatorname{div}\varphi\mathbf{E},$$

since $\operatorname{div}\mathbf{E}=0$ for a field free of charges. The volume integral of $\operatorname{div}\varphi\mathbf{E}$ is transformed into a surface integral. However, according to the meaning attached to L_0 and $\mathscr{E}$, the integration should be performed over the whole region occupied by the field (this is analogous to summation over all the degrees of freedom of the system). At the boundary of this region, the field is equal to zero by definition, so that the surface integral in the expression for energy also becomes zero. From this we obtain the required expression for the energy of an electromagnetic field in the absence of charges:

$$\mathscr{E}=\frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{H}^2)\,dV. \tag{13.21}$$

Hence, the quantity

$$w=\frac{\mathbf{E}^2+\mathbf{H}^2}{8\pi} \tag{13.22}$$

may be interpreted as the energy density of the electromagnetic field. It is invariant with respect to a gauge transformation of a potential.

Conservation of the total energy of field and charges. We shall now show that the energy $\mathscr{E}$ (13.21), together with the energy of the charges contained in the field, is conserved, i.e., $\mathscr{E}$ is the energy in the usual, mechanical, sense of the word, and not some quantity which is formally analogous to it only as regards its derivation from the variational principle.

To do this we multiply equation (12.26) scalarly by **E** and (12.24) scalarly by **H**, and subtract the second from the first. This gives the following relationship:

$$\frac{1}{c}\left(\mathbf{E}\frac{\partial\mathbf{E}}{\partial t}+\mathbf{H}\frac{\partial\mathbf{H}}{\partial t}\right)=\mathbf{E}\operatorname{rot}\mathbf{H}-\mathbf{H}\operatorname{rot}\mathbf{E}-\frac{4\pi\mathbf{jE}}{c}.$$

Now taking advantage of equation (11.29), we reduce the equation obtained to the form

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{H}^2}{8\pi}\right)=-\operatorname{div}\frac{c}{4\pi}[\mathbf{EH}]-\rho\mathbf{vE}. \tag{13.23}$$

Here we have put $\mathbf{j}=\rho\mathbf{v}$ by definition. We now integrate (13.23) over some volume, though not necessarily the whole volume occupied by the field, and transform the integral of div to a surface integral:

$$\frac{\partial}{\partial t}\int\frac{\mathbf{E}^2+\mathbf{H}^2}{8\pi}\,dV=-\int\frac{c}{4\pi}[\mathbf{EH}]\,d\mathbf{s}-\int\rho\,\mathbf{vE}dV. \tag{13.24}$$

Let us first consider the second integral on the right. By definition, the quantity ρdV is the charge element de. The product $\mathbf{E}de$ is the electric force acting on this charge element. The scalar product $de\mathbf{Ev}=de\cdot\mathbf{E}\cdot\frac{d\mathbf{r}}{dt}$ is equal to the work done on the element of charge in unit time or—put in another way—to the change in kinetic energy T of the charge in unit time. Later, we shall show that the magnetic field does not perform work on charges (Sec. 21). To summarize, equation (13.24) can also be written as follows [the last integral in (13.24) will be represented in the form $\frac{dT}{dt}$, i.e., the work done in unit time].

$$\frac{d}{dt}(\mathscr{E}+T)=-\int\frac{c}{4\pi}[\mathbf{EH}]\,d\mathbf{s}. \tag{13.25}$$

The Poynting vector. Thus, the decrease in energy, in unit time, of an electromagnetic field and of the charged particles contained therein is equal to the vector flux $\frac{c}{4\pi}[\mathbf{EH}]$ across the surface bounding the field. If this surface is infinitely distant and the field on it is equal to zero, what we have is simply the energy conservation law for an electromagnetic field and for the charges within it. Otherwise, if the volume is finite, the right-hand side of equation (13.25) indicates what energy passes in unit time through the surface bounding the volume. Hence, the quantity

$$\mathbf{U}=\frac{c}{4\pi}[\mathbf{EH}] \tag{13.26}$$

represents the energy crossing unit area in unit time or, more simply, the energy density flux vector (the Poynting vector).

Field momentum. Similar computations, which we shall not give, show that an electromagnetic field possesses momentum. The momentum of a field is given by the following integral:

$$\mathbf{p}=\int\frac{1}{4\pi c}[\mathbf{EH}]\,dV. \tag{13.27}$$

If the electromagnetic field interacts with some obstacle, for example, the walls of the enclosure in which it is contained, or with a screen, then the momentum of the field is transmitted to the obstacle. The momentum transmitted normally to unit area